

COMBINED PROPERTIES OF FINITE SUMS & FINITE PRODUCTS NEAR ZERO

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ABSTRACT. It was proved that whenever \mathbb{N} is partitioned into finitely many cells, one cell must contain arbitrary length geo-arithmetic progressions. It was also proved that arithmetic and geometric progressions can be nicely intertwined in one cell of partition, whenever \mathbb{N} is partitioned into finitely many cells. In this article we shall prove that similar types of results also hold near zero in some suitable dense subsemigroup S of $((0, \infty), +)$, using the Stone-Čech compactification βS .

1. INTRODUCTION

One of the famous Ramsey theoretic results is van der Waerden's Theorem [11], which says that whenever the set \mathbb{N} of natural numbers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The analogous statement about geometric progressions is easily seen to be equivalent via the homomorphisms $p: (\mathbb{N}, +) \rightarrow (\mathbb{N}, \cdot)$ such that $p(x) = 2^x$, and $q: (\mathbb{N} \setminus \{1\}, \cdot) \rightarrow (\mathbb{N}, +)$, where $q(x)$ is the length of the prime factorization of x .

It has been shown in [1, Theorem 3.11] that any set which is multiplicatively large, that is a piecewise syndetic IP set in (\mathbb{N}, \cdot) must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large geo-arithmetic progressions, that is, sets of the form

$$\{r^j(a + id) : i, j \in \{1, 2, \dots, k\}\}.$$

A well known extension of van der Waerden's Theorem allows one to get the additive increment of the arithmetic progression in the same cell as the arithmetic progression. Similarly for any finite partition of \mathbb{N} there exist some cell A and $b, r \in \mathbb{N}$ such that $\{r, b, br, \dots, br^k\} \subseteq A$. It is proved in [1, Theorem 1.5] that these two facts can be intertwined:

Theorem 1.1. *Let $r, k \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$. Then there exist $s \in \{1, 2, \dots, r\}$ and $a, b, d \in A_s$, such that*

$$\begin{aligned} & \{b(a + id)^j : i, j \in \{0, 1, \dots, k\}\} \cup \{bd^j : j \in \{0, 1, \dots, k\}\} \\ & \bigcup \{a + id : i \in \{0, 1, \dots, k\}\} \subseteq A_s \end{aligned}$$

We know that if $A \subseteq \mathbb{N}$ belongs to every idempotent in $\beta\mathbb{N}$, then it is called an IP^* set. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} , we let $FP(\langle x_n \rangle_{n=1}^\infty)$ be the product analogue of Finite Sum. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} , we say that $\langle y_n \rangle_{n=1}^\infty$ is a

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sum subsystem of $\langle x_n \rangle_{n=1}^\infty$ provided there is a sequence $\langle H_n \rangle_{n=1}^\infty$ of nonempty finite subsets of \mathbb{N} such that $\max H_n < \min H_{n+1}$, and $y_n = \sum_{t \in H_n} x_t$ for each $n \in \mathbb{N}$.

Theorem 1.2. *Let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in \mathbb{N} and A be an IP^* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A.$$

Proof. [2, Theorem 2.6] or see [9, Corollary 16.21]. \square

The algebraic structure of the smallest ideal of βS has played a significant role in Ramsey Theory. It is known that any central subset of $(\mathbb{N}, +)$ is guaranteed to have substantial additive structure. But Theorem 16.27 of [9] shows that central sets in $(\mathbb{N}, +)$ need not have any multiplicative structure at all. On the other hand, in [2] we see that sets which belong to every minimal idempotent of \mathbb{N} , called central* sets, must have significant multiplicative structure.

In case of central* sets a similar result has been proved in [4] for a restricted class of sequences called minimal sequences, where a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} is said to be a minimal sequence if

$$\bigcap_{m=1}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \cap K(\beta\mathbb{N}) \neq \emptyset.$$

Theorem 1.3. *Let $\langle y_n \rangle_{n=1}^\infty$ be a minimal sequence and A be a central* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle x_n \rangle_{n=1}^\infty$ of $\langle y_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A.$$

Proof. [2, Theorem 2.4]. \square

A similar result in this direction in the case of dyadic rational numbers has been proved by Bergelson, Hindman and Leader.

Theorem 1.4. *There exists a finite partition $\mathbb{D} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i.$$

Proof. [3, Theorem 5.9]. \square

In [3], the authors also presented the following conjecture and question.

Conjecture 1.5. *There exists a finite partition $\mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i.$$

Problem 1.6. *Does there exist a finite partition $\mathbb{R} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i?$$

In the section 2, we shall first work on some combined algebraic properties near 0 in the ring of quaternions, denoted by \mathbb{H} . The ring being non abelian, is a division ring having an idempotent 0. In section 3, for any suitable dense subsemigroup S of $((0, \infty), +)$, our aim is to establish partition regularity among two matrices using additive and multiplicative structure of βS , Stone-Ćech compactification of S .

2. COMBINED ALGEBRAIC AND MULTIPLICATIVE PROPERTIES NEAR IDEMPOTENT IN RELATION WITH QUATERNION RINGS

In the following discussion our aim is to extend Theorem 1.2 and Theorem 1.3 for dense subsemigroups $(\mathbb{H}, +)$ in the appropriate context.

Definition 2.1. If S is a dense subsemigroup of $(\mathbb{H}, +)$, we define $0^+(S) = \{p \in \beta S_d : (\forall r > 0)(B_d(r) \in p)\}$.

It is proved in [7], that $0^+(S)$ is a compact right topological subsemigroup of $(\beta S_d, +)$ which is disjoint from $K(\beta S_d)$ and hence gives some new information which are not available from $K(\beta S_d)$. Being compact right topological semigroup $0^+(S)$ contains minimal idempotents of $0^+(S)$. A subset A of S is said to be IP*-set near 0 if it belongs to every idempotent of $0^+(S)$ and a subset C of S is said to be central* set near 0 if it belongs to every minimal idempotent of $0^+(S)$. In [5] the authors applied the algebraic structure of $0^+(S)$ on their investigation of image partition regularity near 0 of finite and infinite matrices. Article [6] used algebraic structure of $0^+(\mathbb{R})$ to investigate image partition regularity of matrices with real entries from \mathbb{R} .

Definition 2.2. Let S be a dense subsemigroup of $(\mathbb{H}, +)$. A subset A of S is said to be an IP set near 0 if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ with $\sum_{n=1}^\infty x_n$ converges such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. We call a subset D of S is an IP* set near 0 if for every subset C of S which is IP set near 0, $C \cap D$ is IP set near 0.

From [10, Theorem 3.2], it follows that for a dense subsemigroup S of $(\mathbb{H}, +)$ a subset A of S is an IP set near 0 if and only if there exists some idempotent $p \in 0^+(S)$ with $A \in p$. Further it can be easily observed that a subset D of S is an IP* set near 0 if and only if it belongs to every idempotent of $0^+(S)$.

Given $c \in \mathbb{H} \setminus \{0\}$ and $p \in \beta \mathbb{H}_d$, the product $c \cdot p$ and $p \cdot c$ are defined in $(\beta \mathbb{H}_d, \cdot)$. One has $A \subseteq \mathbb{H}$ is a member of $c \cdot p$ if and only if $c^{-1}A = \{x \in \mathbb{H} : c \cdot x \in A\}$ is a member of p and similarly for $c \cdot p$.

Lemma 2.3. Let S be a dense subsemigroup of $(\mathbb{H}, +)$ such that $S \cap \mathbb{H}$ is a subsemigroup of $(\mathbb{H} \setminus \{0\}, \cdot)$. If A is an IP set near 0 in S then sA is also an IP set near 0 for every $s \in S \cap B_d(1) \setminus \{0\}$. Further if A is an IP* set near 0 in $(S, +)$ then both $s^{-1}A$ and As^{-1} are IP* set near 0 for every $s \in S \cap B_d(1) \setminus \{0\}$.

Proof. Since A is an IP set near 0 then by [7, Theorem 3.1] there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S with the property that $\sum_{n=1}^\infty x_n$ converges and $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. This implies that $\sum_{n=1}^\infty (s \cdot x_n)$ is also convergent and $FS(\langle sx_n \rangle_{n=1}^\infty) \subseteq sA$. This proves that sA is also an IP set near 0. Similarly we can prove that As^{-1} is also IP set near 0 for every $s \in S \cap B_d(1) \setminus \{0\}$. For the second let A be an IP* set near 0 and $s \in S \cap B_d(1) \setminus \{0\}$. To prove that $s^{-1}A$ is an IP* set near 0 it is sufficient to show that if B is any IP set near 0 then $B \cap s^{-1}A \neq \emptyset$. Since B is an IP set near 0, sB is also an IP set near 0 by the first part of the proof, so that $A \cap sB \neq \emptyset$. Choose $t \in sB \cap A$ and $k \in B$ such that $t = sk$. Therefore $k \in s^{-1}A$ so that $B \cap s^{-1}A \neq \emptyset$. \square

Given $A \subseteq S$ and $s \in S$, $s^{-1}A = \{t \in S : st \in A\}$ and $-s + A = \{t \in S : s + t \in A\}$. In case of product we have to keep in mind the order of elements as the product is noncommutative here.

Definition 2.4. Let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in the ring $(\mathbb{H}, +, \cdot)$, and let $k \in \mathbb{N}$. Then $\text{FP}(\langle x_n \rangle_{n=1}^k)$ is the set of all products of terms of $\langle x_n \rangle_{n=1}^k$ in any order with no repetitions. Similarly $\text{FP}(\langle x_n \rangle_{n=1}^\infty)$ is the set of all products of terms of $\langle x_n \rangle_{n=1}^\infty$ in any order with no repetitions.

Theorem 2.5. Let S be a dense subsemigroup of $(\mathbb{H}, +)$, such that $S \cap B_d(1) \setminus \{0\}$ is a subsemigroup of $(B_d(1) \setminus \{0\}, \cdot)$. Also let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S such that $\sum_{n=1}^\infty x_n$ converges to 0 and A be a IP^* set near 0 in S . Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A.$$

Proof. Since $\sum_{n=1}^\infty x_n$ converges to 0, from [7, Theorem 3.1] it follows that we can find some idempotent $p \in 0^+(S)$ for which $FS(\langle x_n \rangle_{n=1}^\infty) \in p$. In fact $T = \bigcap_{m=1}^\infty \text{cl}_{\beta S_d} FS(\langle y_n \rangle_{n=m}^\infty) \subseteq 0^+(S)$ and $p \in T$. Again, since A is an IP^* set near 0 in S , by the above Lemma 2.3 for every $s \in S \cap B_d(1) \setminus \{0\}$, both $s^{-1}A, As^{-1} \in p$. Let $A^* = \{s \in A : -s + A \in p\}$. Then by [9, Lemma 4.14] $A^* \in p$. We can choose $y_1 \in A^* \cap FS(\langle x_n \rangle_{n=1}^\infty)$. Inductively let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^m, \langle H_i \rangle_{i=1}^m$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following properties:

- (a) $i \in \{1, 2, \dots, m-1\} \Rightarrow \max H_i < \min H_{i+1}$;
- (b) If $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_m} x_t \in A^*$ and $\text{FP}(\langle y_i \rangle_{i=1}^m) \subseteq A$.

We observe that $\{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\} \in p$. Let $B = \{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\}$, let $E_1 = FS(\langle y_i \rangle_{i=1}^m)$ and $E_2 = AP(\langle y_i \rangle_{i=1}^m)$. Now consider

$$D = B \cap A^* \cap \bigcap_{s \in E_1} (-s + A^*) \cap \bigcap_{s \in E_2} (s^{-1}A^*) \cap \bigcap_{s \in E_2} (A^*s^{-1}).$$

Then $D \in p$. Now choose $y_{m+1} \in D$ and $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$. Putting $y_{m+1} = \sum_{t \in H_{m+1}} x_t$ shows that the induction can be continued and this proves the theorem. \square

3. AN APPLICATION OF ADDITIVE AND MULTIPLICATIVE STRUCTURE OF βS

We shall like to produce an alternative proof of the above Theorem 3.1 using additive and multiplicative structure of βS . We need the following notion.

Theorem 3.1. Let $u, v \in \mathbb{N}$. Let M be a finite image partition regular matrix over \mathbb{N} of order $u \times v$, and let N be an infinite image partition regular near 0 matrix over a dense subsemigroup S of $((0, \infty), +)$. Then

$$\begin{pmatrix} M & O \\ O & N \end{pmatrix}$$

is image partition regular near 0 over S .

Definition 3.2. Let S be a subsemigroup of $((0, \infty), +)$ and let A be a finite or infinite matrix with entries from \mathbb{Q} . Then $I(A) = \{p \in 0^+ : \text{for every } P \in p, \text{ there exists } \vec{x} \text{ with entries from } S \text{ such that all entries of } A\vec{x} \text{ are in } P\}$.

The following lemma can be easily proved as [8, Lemma 2.5].

Lemma 3.3. Let A be a matrix, finite or infinite with entries from \mathbb{Q} .

- (a) The set $I(A)$ is compact and $I(A) \neq \emptyset$ if and only if A is image partition regular near 0.

(b) If A is finite image partition regular matrix, then $I(A)$ is a sub-semigroup of $(0^+, +)$.

Next, we shall investigate the multiplicative structure of $I(A)$. In the following Lemma 3.4, we shall see that if A is a image partition regular near 0 then $I(A)$ is a left ideal of $(0^+, \cdot)$. It is also a two-sided ideal of $(0^+, \cdot)$, provided A is a finite image partition regular near 0.

Lemma 3.4. *Let A be a matrix, finite or infinite with entries from \mathbb{Q} .*

(a) *If A is an image partition regular near 0, then $I(A)$ is a left ideal of $(0^+, \cdot)$.*

(b) *If A is a finite image partition regular near 0, then $I(A)$ is a two-sided ideal of $(0^+, \cdot)$.*

Proof. (a). Let A be a $u \times v$ image partition regular matrix, where $u, v \in \mathbb{N} \cup \{\omega\}$. Let $p \in 0^+$ and $q \in I(A)$. Also let $U \in p \cdot q$. Then $\{x \in S : x^{-1}U \in q\} \in p$. Choose $z \in \{x \in S : x^{-1}U \in q\}$. Then $z^{-1}U \in q$. So there exists \vec{x} with entries from S such that $y_j \in z^{-1}U$ for $0 \leq j < u$ where $\vec{y} = A\vec{x}$,

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} \text{ and } \vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}; \vec{y} \text{ and } \vec{x} \text{ are } u \times 1 \text{ and } v \times 1 \text{ matrices respectively.}$$

Now $y_i \in z^{-1}U$ for $0 \leq i < u$ implies that $zy_i \in U$ for $0 \leq i < u$. Let $\vec{x}' = z\vec{x}$ and $\vec{y}' = z\vec{y}$. Then $\vec{y}' = A\vec{x}'$. So there exists \vec{x}' with entries from S such that all entries of $A\vec{x}'$ are in U . Therefore $p \cdot q \in I(A)$. So $I(A)$ is a left ideal of $(0^+, \cdot)$.

(b) Let A be a $u \times v$ matrix, where $u, v \in \mathbb{N}$. By previous lemma, $I(A)$ is a left ideal. We now show that $I(A)$ is a right ideal of $(0^+, \cdot)$. Let $p \in \beta S$ and $q \in I(A)$. Now let $U \in q \cdot p$. Then $\{x \in S : x^{-1}U \in p\} \in q$. So there exists \vec{x} with entries in

S such that $y_i \in \{x \in S : x^{-1}U \in p\}$ for $0 \leq i < u$, where $\vec{y} = A\vec{x}$, $\vec{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_{u-1} \end{pmatrix}$

and $\vec{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_{v-1} \end{pmatrix}$; \vec{y} and \vec{x} are $u \times 1$ and $v \times 1$ matrices respectively. Now for

$0 \leq i < u, y_i \in \{x \in S : x^{-1}U \in p\}$. Hence $y_i^{-1}U \in p$ for $0 \leq i < u$. This implies $\bigcap_{i=0}^{u-1} y_i^{-1}U \in p$. So $\bigcap_{i=0}^{u-1} y_i^{-1}U \neq \emptyset$. Let $z \in \bigcap_{i=0}^{u-1} y_i^{-1}U$. Therefore $z \in y_i^{-1}U$ for all $i \in \{0, 1, 2, \dots, u-1\}$. Hence $y_i z \in U$ for $0 \leq i < u$. Let $\vec{x}' = \vec{x}z$ and $\vec{y}' = \vec{y}z$. Then $\vec{y}' = A\vec{x}'$. So there exists \vec{x}' with entries from S such that all entries of $A\vec{x}'$ are in U . Thus $q \cdot p \in I(A)$. Therefore $I(A)$ is also a right ideal of $(\beta\mathbb{N}, \cdot)$. Hence $I(A)$ is a two-sided ideal of $(0^+, \cdot)$. \square

Alternative proof of Theorem 2.2.7. Let $r \in \mathbb{N}$ be given and $\epsilon > 0$. Let $\mathbb{Q} = \bigcup_{i=1}^r E_i$. Suppose that A be a $u \times v$ matrix where $u, v \in \mathbb{N}$. Also let $A = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$. Now by previous Lemma 3.4, $I(M)$ is a two-sided ideal of $(0^+, \cdot)$. So $K(0^+, \cdot) \subseteq I(A)$. Also by same Lemma 3.4, $I(N)$ is a left ideal of $(0^+, \cdot)$. Therefore

$K(0^+, \cdot) \cap I(N) \neq \emptyset$. Hence $I(M) \cap I(N) \neq \emptyset$. Now choose $p \in I(M) \cap I(N)$. Since $\mathbb{Q} = \bigcup_{i=1}^r E_i$, there exists $k \in \{1, 2, \dots, r\}$ such that $E_k \in p$. Thus by definition of $I(M)$ and $I(N)$, there exist $\vec{x} \in S^v$ and $\vec{y} \in S^\omega$ such that $M\vec{x} \in E_k^u$ and $N\vec{y} \in E_k^\omega$. Take $\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$. Then $A\vec{z} = \begin{pmatrix} M\vec{x} \\ N\vec{y} \end{pmatrix}$. So $A\vec{z} \in E_k^\omega$. Therefore $A = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ is image partition regular near 0. \square

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